

# The impact of memristive devices and systems on nonlinear circuit theory

Ricardo Riaza

*Abstract*—In this talk we present a discussion of the impact of memristive devices (memristors, memcapacitors and meminductors) and memristive systems on the fundamentals of nonlinear circuit theory and also on electronics. The interest on this topic has increased continuously since the design in 2008 of a nanoscale device with a charge-flux characteristic, displaying a very promising industrial scope in memory design. We survey theoretical concepts and also discuss some recent applications in this area, providing references for further study.

*Keywords*—Nonlinear circuit, memristor, memcapacitor, meminductor, nonlinear oscillator, chaotic circuit, resistive memory, memristive neural network.

## I. INTRODUCTION

THE history of memristive devices can be traced back to the seminal work of Leon Chua, who in 1971 postulated the existence of a nonlinear device whose characteristic would be defined by a charge-flux relation [1]. This device would be the fourth basic circuit element, besides the resistor, the inductor and the capacitor which relate the voltage-current, current-flux and voltage-charge pairs, respectively. The report in 2008 of a nanometer-scale device displaying a memristive characteristic [2] has had a great impact in the electrical and electronic engineering communities and has raised a renewed interest towards these devices.

The memristor and related devices are likely to play a relevant role in electronics in the near future, especially at the nanometer scale. Many applications are already reported, e.g. in pattern recognition, memory design, signal processing, design of nonlinear oscillators and chaotic systems, adaptive systems, etc. (see [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30]). Commercial memory chips based on the memristor are expected to be released in the near future. The notion of a device with memory was extended to the reactive setting by Di Ventra, Pershin and Chua [8] in order to define *memcapacitors* and *meminductors*.

We survey in this talk some basic properties of memristors, memcapacitors and meminductors (cf. Section II) together with those of the so-called *memristive systems*, introduced by Chua and Kang in [31] (Section III). The impact of such devices and systems on the fundamentals of nonlinear circuit theory and electronics is surveyed in Section IV, whereas Section V compiles some applications of these devices and systems in electronics. Concluding remarks are compiled in Section VI.

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## II. MEMRISTORS, MEMCAPACITORS AND MEMINDUCTORS

### A. Memristors

The memristor is a nonlinear device defined by a charge-flux characteristic, which may be have either a charge-controlled or a flux-controlled form. In a charge-controlled setting, the characteristic reads as

$$\varphi = \phi(q), \quad (1)$$

for some  $C^1$  map  $\phi$ . The incremental *memristance* is

$$M(q) = \phi'(q).$$

Using the relations  $\varphi' = v$ ,  $q' = i$  we get the voltage-current characteristic

$$v = M(q)i. \quad (2)$$

This relation shows that the device behaves as a resistor in which the resistance depends on  $q(t) = \int_{-\infty}^t i(\tau)d\tau$ , hence the *memory-resistor* name. This is the key feature of the device. In greater generality, one may consider (2) as a particular case of a fully nonlinear characteristic of the form

$$v = \eta(q, i).$$

In turn, a flux-controlled memristor has a characteristic of the form

$$q = \xi(\varphi), \quad (3)$$

and the incremental *memductance* is

$$W(\varphi) = \xi'(\varphi).$$

The voltage-current relation has in this case the form

$$i = W(\varphi)v \quad (4)$$

or, in a fully nonlinear context,

$$i = \zeta(\varphi, v).$$

A memristor governed by (2) or (4) is said to be *strictly locally passive* if  $M(q) > 0$  or  $W(\varphi) > 0$  for all  $q$  or  $\varphi$ , respectively. In the presence of coupling effects (if eventually displayed), this requirement must be restated by asking the memristance or memductance matrices to be positive definite.

### B. Memcapacitors and meminductors

Di Ventra, Pershin and Chua extended in [8] the idea of a device with memory to reactive elements. A (voltage-controlled) *memcapacitor* has a characteristic of the form

$$q = C_m(\varphi)v. \quad (5)$$

Here  $C_m$  is the *memcapacitance*. The distinct feature of this device is that the memcapacitance depends on the state variable  $\varphi(t) = \int_{-\infty}^t v(\tau)d\tau$ , so that the relation  $q(t) = C_m(\int_{-\infty}^t v(\tau)d\tau)v(t)$  reflects the device history. Analogously, a (current-controlled) *meminductor* is governed by

$$\varphi = L_m(q)i, \quad (6)$$

and  $L_m(q)$  is the *meminductance*, which reflects the device history via the variable  $q$ .

A fully nonlinear formalism for voltage-controlled memcapacitors and current-controlled meminductors is obtained after replacing (5) and (6) by the characteristics

$$q = \omega(\varphi, v) \quad (7)$$

and

$$\varphi = \theta(q, i), \quad (8)$$

respectively, for certain maps  $\omega, \theta$ .

### III. MEMRISTIVE SYSTEMS

Chua and Kang extended in [31] the ideas underlying the notion of a memristor to define a *memristive system*. A current-controlled memristive system is defined by a relation of the form

$$q' = \mu(q, i) \quad (9a)$$

$$v = M(q, i)i, \quad (9b)$$

whereas a voltage-controlled one is defined by

$$\varphi' = \psi(\varphi, v) \quad (10a)$$

$$i = W(\varphi, v)v. \quad (10b)$$

In particular, a charge-controlled memristor is an instance of (9) in which  $q$  represents charge,  $\mu(q, i)$  amounts to the current  $i$  and  $M$  does not depend on  $i$ , as discussed in Section II. In this case,  $M(q)$  is the (incremental) memristance, as indicated above; this arises as the derivative of a nonlinear flux-charge relation  $\varphi = \phi(q)$ , and this supports the “charge-controlled” term. Similarly, a flux-controlled memristor is a particular instance of (10) in which  $\varphi$  stands for the magnetic flux,  $\psi(\varphi, v)$  amounts to the voltage  $v$  and  $W$  does not depend on  $v$ . Now  $W(\varphi)$  is the (incremental) memductance. Note that, for the sake of notational simplicity, we use  $q$  and  $\varphi$  to represent the dynamic variables of general memristive devices, although only for memristors in strict sense these variables denote charges and fluxes, respectively.

Other instance of memristive devices are thermistors, discharge tubes and ionic systems. In all these systems (and all in “classical” memristors), the matrices  $M$  and  $W$  are actually

independent of  $i$  and  $v$ , respectively, yielding the simpler forms

$$q' = \mu(q, i) \quad (11a)$$

$$v = M(q)i \quad (11b)$$

in the current-controlled context, and

$$\varphi' = \psi(\varphi, v) \quad (12a)$$

$$i = W(\varphi)v \quad (12b)$$

in a voltage-controlled setting.

Many recent applications of memristive devices are better framed in the context defined by (9) and (10) (or in the simplified contexts represented by (11) and (12)) rather than in the setting defined in Section II. This is also the case for memcapacitive and meminductive systems, as discussed in [8].

### IV. HIGHER ORDER DEVICES AND THE FUNDAMENTALS OF NONLINEAR CIRCUIT THEORY

Going back to the context defined by memristors, memcapacitors and meminductors (cf. Section II), it is worth noticing that both (5) and (6) come from differentiating a two-variable relation, namely  $\sigma = \alpha(\varphi)$  for voltage-controlled memcapacitors and  $\rho = \beta(q)$  for current-controlled meminductors; here  $\sigma$  and  $\rho$  arise as the time-integrals of  $q$  and  $\varphi$ , respectively. By using the differentiated relations (5) and (6) we get rid of these second order variables. This is not the case for so-called *second-order devices*, for which either  $\sigma$  or  $\rho$  appear explicitly in the memcapacitance or the meminductance. Specifically, a *charge-controlled memcapacitor* is a device defined by the relations

$$\sigma' = q \quad (13a)$$

$$q' = i \quad (13b)$$

$$v = C^{-1}(\sigma)q, \quad (13c)$$

whereas a *flux-controlled meminductor* is characterized by

$$\rho' = \varphi \quad (14a)$$

$$\varphi' = v \quad (14b)$$

$$i = L^{-1}(\rho)\varphi. \quad (14c)$$

Noteworthy, the relations (13c) and (14c) arise as the differentiated form of certain mappings  $\varphi = \gamma(\sigma)$  and  $q = \delta(\rho)$ , via the relations  $\sigma' = q$ ,  $\rho' = \varphi$ . In a natural way this leads to other second order devices, such as those relating  $\sigma$  and  $\rho$  (cf. [5]). And in a fully nonlinear formalism, we may replace (13c) and (14c) by characteristics of the form

$$v = \nu(\sigma, q), \quad (15)$$

and

$$i = \chi(\rho, \varphi), \quad (16)$$

respectively.

In greater generality, one may look at the mathematical formalism of nonlinear circuit theory as a framework in which the basic circuit variables  $v$  (voltage) and  $i$  (current), together with some integral variables (not only  $\sigma$  and  $q$  but

eventually higher order ones as well), are interrelated to define a dynamical system restricted only by Kirchhoff laws, which remain at (and actually define) the core of circuit theory. This general point of view is developed in [32], [33] and the reader is referred to these papers for details. See also [34] and the above-referenced paper [5].

## V. SOME APPLICATIONS

Many applications of memristive devices are reported in the design of nonlinear oscillators and chaotic circuits; we refer readers interested on this application area to the papers [4], [6], [9], [11], [12], [20], [21], [22], [27], [29] and references therein. Memristors are very promising in non-volatile memory design and in the implementation of resistive random access memory (RRAM or ReRAM); some recent references in this direction are [30], [35], [36], [37], [38], [39]. There is already a vast amount of literature on memristive neural networks; cf. [16], [23], [25], [40], [41], [42], [43], [44], [45], [46], [47], [48], [49], [50], [51]; this research activity stems from the fact that the memristor provides a very natural way to implement a synapse, displaying two stable states which are easily adjustable using electrical pulses. This approach paves the way for an efficient hardware implementation of artificial neural networks. Other actual or potential applications of memristors are discussed in [3], [5], [7], [8], [10], [13], [14], [15], [17], [18], [19], [24], [28], [52].

## VI. CONCLUDING REMARKS

Research on memristors and memristive devices has exploded in the last seven years, following the announcement by HP of the design of a nanoscale memristor in 2008. Traditional points of view on the fundamentals of circuit theory are being revisited and a lot of applications are being reported. Many analytical properties of memristive circuits remain to be solved, though, and much more applications are yet to come. This communication reports some recent research directions in this area, and the reader is referred to the references compiled below for a more detailed introduction to the main features and applications of memristive devices and systems.

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